Logarithms: A Tutorial

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Abstract. What is are logarithms (logs) and why should doctors need to know about them? The answer lies in the history of computation and the first steps towards making medicine scientific. Before digital computation, logs provided a quick way of computing the multiplication of complicated products, by transforming these complicated products into simple sums. Logs were computational shortcuts when the user required ratios to effect the computation. Logs also allowed different measurements to be compared, especially across large scales. These scales persist in medicine today, especially in nuclear medicine and internal medicine. This article defines the concepts used in developing the logarithm rigorously. It places the development of the logarithm in historical context and shows an example of why they were useful in pre-computer eras. I finish the article showing a large series of examples of the use of logarithms in medical situations.

0. A problem of measurement

Scientific modern medicine is concerned with measurement. Medical doctors often wish to compare two different methods of measuring some quantity such as blood pressure, gestational age, cardiac stroke volume, and so on. Frequently though, beyond the usual calibration and repeatability problems, which we are not interested in here, we cannot generally regard any method of measurement as producing a definitive point estimate of the phenomenon under study. The patient's exact blood pressure is never known, for example. Only an approximation to that idealized quantity is known, that is changing over time. In any case, in many parts of medicine, one does not care what the exact point estimate is: one wishes only to measure changes in these measurements as exactly as possible: did the patient's blood pressure increase or decrease overnight after drug x was administered? These are the questions for which ratios, and their natural extension, logarithms, were invented for.

This short article does three things. It defines the concepts used in developing the logarithm rigorously. It places the development of the logarithm in historical context and shows an example of why they were useful in pre-computer eras. I finish the article showing a large series of examples of the use of logarithms in medical situations.

0.1 Defining terms

Before I answer the question posed in the introduction, I'd better define the concepts used herein. Because the logarithm is a mathematical property, it is important to define terms precisely, before going on to use the concept in applied medical situations.
Definition 0. Power. A power is an exponent to which a given quantity is raised. The expression \( x^a \) is so, known as \( x \) to the \( a \)-th power. A number of powers of \( x \) are plotted in figure 1 below (cf. Derbyshire 2004, pp. 68 and 73). The code for generating the figures is locked inside the (code) brackets. It is available in the appendix.

![Figure 1: Different powers of some variable x.](image)

Definition 1. Logarithm. The logarithm \( \log_b x \) for a base \( b \) and a number \( x \) is defined as the inverse function of taking \( b \) to the power \( x \), i.e. \( b^x \). So, for any \( x \) and \( b \),

\[
x = \log_b (b^x),
\]

or equivalently,

\[
x = b^{\log_b x}.
\]

We obtain \( x = b^{\log_b x} \) by using the anti-log or exponential function, \( e^x \). Now we need to define base and inverse function, which I'll do in a moment. Basically, the principle is to express a number as another number with an index or exponent, so \( 100 = 10^2 \), \( 16 = 2^4 \). The ‘another number’ here is the base of the logarithm. Most of the time, we'll see bases of 2, 10, and the natural base of \( e \), where \( e = 2.71828 \).

Definition 2. Base. The word "base" in mathematics is used to refer to a particular mathematical object that is used as a building block. The most common uses are the related concepts of the number system, whose digits are used to represent numbers and the separate but related number system in which logarithms are defined. 'Base' can also be used to refer to the bottom edge or surface of a geometric figure. A real number \( x \) can be represented using any integer number \( b \neq 0 \) as a base (sometimes also called a radix or scale). The choice of a base gives us a representation of numbers known as a number system.

- Example: Bases of ten

The number ten (10) is a collection of ones, a distance of ten 1's down the number line from 0, where the number line starts. It can be represented in various bases as
10 = 1010₂
   = 101₃
   = 22₄
   = 20₅
   = 14₆
   = 13₇
   = 12₈
   = 11₉
   = 10₁₀

Why? Because

10 = 1 × 2³ + 1 × 2¹
   = 1 × 3³ + 1 × 3⁰
   = 2 × 4¹ + 2 × 4⁰
   \vdots

and so on.

The base of a logarithm is a number \(b\) used to define the number system in which the logarithm is computed. In general, the logarithm of a number \(x\) in base \(b\) is written \(\log_b x\). The symbol \(\log\) is an abbreviation regrettably used both for the common logarithm \(\log_{10} x\) (by doctors, engineers and physicists and indicated on pocket calculators) and for the natural logarithm \(\log_e x\) (by mathematicians). ln(x) denotes the natural logarithm \(\log_e x\) (See Weisstein, 2006 for details of this).

This says: the logarithm to the base \(a\) of a variable \(x\) is the exponent \(b\) to which \(a\) must be raised to yield the quantity \(x\) (Gullson, 1997, pg. 354), making these two equations equivalent:

\[
x = a^b; \quad \log_a x = b.
\]  
(3)

Just remember the relationship described in equation (3) above, and most of the time you'll be fine. The table below plots the relationship described in equation (3) for a range of values.

<table>
<thead>
<tr>
<th>Number</th>
<th>Exponential Expression</th>
<th>Logarithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>10³</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>10²</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>10¹</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>10⁰</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>10⁻¹</td>
<td>-1</td>
</tr>
<tr>
<td>0.01</td>
<td>10⁻²</td>
<td>-2</td>
</tr>
<tr>
<td>-0.0001</td>
<td>10⁻³</td>
<td>-3</td>
</tr>
</tbody>
</table>

Table 1: Some simple examples of logs in action. Observe that they can be negative.

Why do we analyze the logs of measurements rather than the logs themselves? This question goes straight to what logs are, why they were invented in the first place, and what their role is in modern medicine. The proper place to answer this question is in the history of the logarithm concept.
1. History of logarithms

The logarithm was invented by John Napier, a Scottish lord and mathematician, working alone in a castle near Edinburgh. His *Mirifici Logarithmorum Canonis Constructio* (available in its full text at http://johnnapier.com/table_of_logarithms_001.htm) was posthumously published in 1619. In this work, he also invented the use of the decimal point when writing fractions. Lord Moulton, (quoted in Gladstone-Millar, 2003: 36), remarking on the 300th anniversary of Napier's discovery of the logarithm, wrote:

No previous work had led up to it, nothing had foreshadowed it or heralded its arrival. It stands isolated, breaking upon human thought abruptly, without borrowing from the works of other intellects or following known lines of mathematical thought.

The whole point of a logarithm is to ease the calculation of square roots and cubes by multiplication and division. The logarithm is a computational tool which facilitates use of powers when you're computing some product by hand. To appreciate just how useful this is, an example is apposite.

- Example: Logarithms and simple by-hand multiplication

Let's see how hard that was back in the day before the invention of logs.

Imagine we want to multiply some quantities, say 7464 × 243.

We'd do it like this:

\[
\begin{array}{c}
7464 \\
\times 243 \\
\hline
1492800 \\
+298560 \\
+22392 \\
\hline
1813752.
\end{array}
\]

Note the number of places where error can creep in: twice when the multiplication is carried out, twice when the sums are added, and again for the second line. All in all, five steps where error can creep in. Napier simplified this considerably when he invented the technique of taking logs. He produced the first tables and gave the world one of its first calculating engines: Napier's Bones.

Napier's method of working out the same product is as follows. First, work out the characteristics of the number you want to multiply. The characteristic is always one digit less than the number of digits. So, for the number 7464, for example, the characteristic is 3. The mantissa is found using log tables, which can be easily found online (see http://wwwchem.uwimona.edu.jm:1104/courses/ph/log10tab.html for example ). For 7464, locate the 74th column and the 6th row. This gives you 0.8727. There remains the unit figure of 4, and for this you go further along the 74 row to the ADD column looking for the number 4. This number is 2, which you add to the mantissa, giving an overall mantissa of 0.8729. Because the characteristic is 3, and the mantissa is 0.8729, the complete logarithm of 7464 is therefore 3.8729. The same process is applied for 243, giving a logarithm of 2.3856. We can go to any level of accuracy desired with this calculation. Table 2 below compares the calculations side by side. It is obvious which method is more efficient.

<table>
<thead>
<tr>
<th>number</th>
<th>log</th>
</tr>
</thead>
<tbody>
<tr>
<td>7464</td>
<td>3.8729</td>
</tr>
<tr>
<td>× 243</td>
<td>+2.3856</td>
</tr>
<tr>
<td></td>
<td>6.2585</td>
</tr>
</tbody>
</table>

*Table 2: Multiplication of numbers becomes addition of logs.*

Now that we have the logarithmic product, we can change 6.2585 back to a regular number by employing anti-logarithms. We can take advantage of the relationship described in equation (3).
Let's take the mantissa, \((0.2585)\), first. We use an antilogarithm table to make the calculation (see http://wwwchem.uwimona.edu.jm:1104/courses/pH/antilog.html for examples).

Locate the 25th row and look under the 8. This gives 0.1811. Then go to the ADD column for the number 5, and find 2. Adding 2 to 0.1811 gives 0.1813. The characteristic of 6 must now have an extra digit added, so the final number will have seven digits. We do this by adding zeros. The answer we get is 1,813,000, which is very close to the true answer of 1813752, and we are only using 4-digit precision. With increasingly detailed logarithmic tables (ten digits, say) the result would be much closer to perfect accuracy.

Once the technique is mastered, the complex multiplication required in practical calculations become simple addition. This is the power of the logarithmic technique, and it explains why it was used in the development of modern medicine.

2. Uses of Logs in Medicine

Logs fulfil a need in modern medicine to compare one quantity with another. Here are a series of examples to show their usefulness as well as their ubiquity.

- **Example: Radioactive Decay**

A radioactive substance loses mass as time passes. Let's say it's mass, \(m\), is related to the time elapsed, \(t\), by an equation of the form

\[
y = ae^{-kt}, \quad (k > 0) .
\] (4)

Now, at time \(t = 0\), the initial mass described by equation (2) will be

\[
y = ae^{-k(0)} \quad = a \times 1 \quad = a
\]

The letter \(k\) here represents a constant associated with the specific radioactive substance being considered. If, for a particular radioactive substance, the initial mass is \(a = 1000\) grams and \(k = 0.70\), then equation (2) will look like this:

\[
y = 1000 \, e^{-0.70 \cdot t} .
\] (5)

Now we can relate the substance's mass \(y\) (in grams) with the time elapsed, \(t\), (in minutes). So at time \(t = 1\) (i.e. after 1 minute has gone by), the mass is

\[
y = 1000 \, e^{-0.70} \quad \approx 1000 \, (0.496585) \quad \approx 496.59 \text{ grams}.
\]

The graph of \(y = 1000 \, e^{-0.70 \cdot t}\) is illustrated in the figure (2) below.
Example: Bacteria growth

A certain type of bacteria, escheridia roisin mcnamariens, triples its numbers each day. Initially, there were 500,000 such bacteria present. Let $y$ represent the number of bacteria present and $t$ represent the number of days elapsed.

1. Find the equation that relates $y$ to $t$

$$y = 500000 \cdot e^{3t}.$$  \hfill (6)

2. What is the graph of the equation you've found?

3. How many bacteria are present after 4 days?
4. How many bacteria are present after 6 days?

\[
\text{N}[5 \times 10^6] = 3.283 \times 10^8
\]

Here are some problems that are solved in the same way using the same tools.

**Example:** Maintenance of Neonatal Giraffe Beds

The annual maintenance cost, \( y \), of a Giraffe Ominbed is related to the number of years it is run, \( t \), by the equation

\[
y = 1000 e^{0.05t} \cdot (\forall t \geq 0)
\]

When is the likely decommissioning date for the bed?

**Example:** pH problems

\[
pH = -\log(\text{hydrogen ion concentration}) = -\log[H^+].
\]

Many applications of pH scales exist in medicine. The usual questions asked are: what is the pH concentration in an aqueous solution with pH = 13.22? We can use log rules to calculate this very easily.

**Example:** Hormone levels in pregnant women

When a woman becomes pregnant, the hormone HCG (human chorionic gonadotropin) is produced in order to enable the baby to develop. During the first few weeks of pregnancy, the level of HCG hormone grows exponentially, starting with the day the embryo is implanted in the womb. However, the rate of growth varies with each pregnancy. Therefore, doctors cannot use just a single test to determine how long a woman has been pregnant. Commonly, the HCG levels are measured two days apart to look for this rate of growth. A woman who is not pregnant will often have an HCG level of between 0 and 5 mIU (milli-international units) per ml (milliliter).

If we have a patient whose HCG level was 200 mIU/ml on March 21, while two days later her HCG level was 392 mIU/ml, what equation models the growth of HCG for this mother's pregnancy? And assuming the mother's HCG level was 5 mIU/ml, how many days after the implantation was the mother's first (March 21st) visit?

### 3. Logarithms in 21st Century Medicine

Medicine continues to be about the measurement of quantities over time, and trying to affect changes in those quantities. Since the log is a tool to aid this practice, perhaps their time has not yet passed.
References & Links


Appendix

(Code for figure 1)

(* turn off various messages *) Off[General::spell];
Off[General::spell1];
Off[General::"ovfl"]; Off[General::"unfl"];  
Off[Power::infy]; Off[Infinity::"indet"];  
Off[ParametricPlot::pptr];
Off[Graphics3D::"nlist3"];  
Off[NDSolve::mxst];
colMax = 0.79;  
red = RGBColor[1, 0, 0];  
blue = RGBColor[0, 0, 1];
yellow = RGBColor[0, 1, 0];
green = RGBColor[0., 0.392193, 0.];
violet = RGBColor[0.580401, 0., 0.827494];
darkGreen = RGBColor[0., 0.392193, 0.];

Show[Plot[Evaluate@Table[x^n, {n, -3, 3}]],
Evaluate[{{x, Sequence @@ #}}, DisplayFunction -> Identity,
PlotStyle -> {red, yellow, blue, darkGreen, violet,
darkGreen, red}], AxesLabel -> TraditionalForm /@ {x, x^n},
PlotRange -> {Automatic, {-10, 10}}] & /@ {{-3, 0}, {0, 3}},
DisplayFunction -> $DisplayFunction];

(code for figure 2)

Plot[1000 e^-.70 t, {t, 0, 10}, PlotStyle -> {red},
AxesLabel -> TraditionalForm /@ {t, y}];

(code for figure 3)

Plot[5 e^3 t, {t, 0, 1}, PlotStyle -> {red},
AxesLabel -> TraditionalForm /@ {t, y}];